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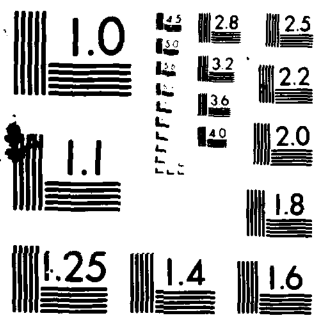

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A NEW APPROACH TO EULER SPLINE. I

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A NEW APPROACH TO EULER SPLINE. I

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ABSTRACT

Starting from the exponential Euler polynomials discussed by Euler in [1], the author introduced in [2] the so-called exponential Euler splines. Here we describe a new approach to these splines. Let t be a constant such that

$$t = |t|e^{i\alpha}, \quad -\pi < \alpha < \pi, \quad t \neq 0, \quad t \neq 1 .$$

Let $S_1(x;t)$ be the cardinal linear spline such that

$$S_1(\nu;t) = t^\nu \text{ for all } \nu \in \mathbf{Z} .$$

Starting from $S_1(x;t)$ it is shown that we obtain all higher degree exponential Euler splines recursively by the averaging operation

$$S_n(x;t) = \int_{x-1/2}^{x+1/2} S_{n-1}(u;t)du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u;t)du \quad (n = 2,3,\ldots) .$$

Here $S_n(x;t)$ is a cardinal spline of degree n if n is odd, while $S_n(x+1/2;t)$ is a cardinal spline if n is even. It is shown that they have the properties

$$S_n(\nu;t) = t^\nu \text{ fo. } \nu \in \mathbf{Z} ,$$

$$\lim_{n\to\infty} S_n(x;t) = t^x = |t|^x e^{i\alpha x} .$$

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SIGNIFICANCE AND EXPLANATION

Let t be a real or complex constant, which we assume not to be negative, also $t \neq 0$ and $t \neq 1$. Let $S_1(x;t)$ denote the function of x obtained by linear interpolation of the points (ν, t^ν) (ν running over all integers). It is shown that we obtain all <u>exponential Euler splines</u> $S_n(x) = S_n(x;t)$, introduced in [2], recursively, by starting from $S_1(x,t)$, and applying the recursive formula

$$S_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S_{n-1}(u)du \Big/ \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{n-1}(u)du, \qquad (n = 2,3,\ldots) .$$

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NEW APPROACH TO EULER SPLINES. I

I. J. Schoenberg

(cont'd to p. A)

1. Introduction. Here I wish to describe a new recursive construction of these exponential Euler splines which I introduced in [2], a construction to be described in §2. We first recall their definition as given in [2].

Let $S_n = \{S(x)\}$ denote the class of cardinal splines $S(x)$ of degree n, having knots at the integers ν. Furthermore, let $S^* = \{S(x);\ S(x+\frac{1}{2}) \in S_n\}$ be the class of midpoint cardinal splines, i.e. having knots at $\nu + \frac{1}{2}$.

For a constant t such that

(1.1) $$t = |t|e^{i\alpha}, \quad -\pi < \alpha < \pi, \quad t \neq 0, \quad t \neq 1\ ,$$

we wish to construct the cardinal spline which interpolates at all integers the exponential function $t^x = |t|^x e^{i\alpha x}$. In [•] this problem was solved within each of the two classes S_n and S_n^*. These solutions $s_n(x)$ and $s_n^*(x)$ were constructed as follows.

Definition of $s_n(x) \in S_n$. We define $s_n(x)$ $(n \geq 1)$ in two steps. We start from Euler's generating function [1, Chap. VII, §178]

(1.2) $$\frac{t-1}{t-e^z}\, e^{xz} = \sum_0^\infty \frac{A_n(x;t)}{n!}\, z^n\ ,$$

defining the Eulerian polynomial $A_n(x;t)$, set

(1.3) $$p_n(x) = A_n(x;t)/A_n(0;t)\ ,$$

and define

(1.4) $$s_n(x) = p_n(x) \quad \text{if} \quad 0 \leq x < 1\ .$$

The second step consists in extending the definition of $s_n(x)$ to all real x by means of the functional equation

(1.5) $$s_n(x+1) = ts_n(x)\ .$$

The remarkable property of the polynomial $p_n(x)$, defined by (1.3), is that the

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resulting $s_n(x) \in C^{n-1}(\mathbb{R})$ and therefore $s_n(x) \in S_n$. From (1.3) $p_n(0) = 1$ and now (1.5) shows that $s_n(\nu) = t^\nu$ for all $\nu \in \mathbb{Z}$. The function $s_n(x)$ is the exponential Euler spline of the class S_n. Euler provided in 1755 the seed (1.3); all that I had to do in [2] to obtain $s_n(x)$, was to extend this seed $p_n(x)$ to all x by means of (1.5).

Definition of $s_n^*(x) \in S_n^*$. Now we use the generating function

$$\frac{t-1}{t-e^z} e^{(x+1/2)z} = \sum_0^\infty \frac{B_n(x;t)}{n!} z^n , \tag{1.6}$$

defining the midpoint Eulerian polynomials $B_n(x;t)$, set

$$p_n^*(x) = B_n(x;t)/B_n(0;t) , \tag{1.7}$$

and define

$$s_n^*(x) = p_n^*(x) \quad \text{if} \quad -\frac{1}{2} \leq x < \frac{1}{2} . \tag{1.8}$$

The second step is to extend $s_n^*(x)$ to all real x by

$$s_n^*(x+1) = ts_n^*(x) . \tag{1.9}$$

Now (1.7) is such that $s_n^*(x) \in C^{n-1}(\mathbb{R})$ and therefore $s_n^*(x) \in S_n^*$.

In §2 we define recursively, by (2.1) and (2.5), a sequence of cardinal splines

$$S_n(x) = S_n(x;t) \quad (n = 1,2,\ldots) , \tag{1.10}$$

such that

$$S_n(x) = \begin{cases} s_n(x) & \text{if } n \text{ is odd} , \\ s_n^*(x) & \text{if } n \text{ is even} . \end{cases} \tag{1.11}$$

In the remainder of the paper we use only the new construction to obtain all known properties of the Euler splines and also some new ones.

While our assumptions (1.1) excluded negative values of t, we use in §6 the new construction to discuss the classical case when $t = -1$.

2. A new approach to the exponential Euler splines. As t is kept fixed throughout, we simplify our notation by writing $S_n(x;t) = S_n(x)$. We define $S_1(x)$ by the conditions

(2.1) $$S_1(x) \in \mathcal{S}_1, \quad S_1(\nu) = t^\nu \quad (\nu \in \mathbb{Z}) .$$

Plotting the sequence (t^ν) in the complex plane we see that the graph of $z = S_1(x)$ $(-\infty < x < \infty)$ is the biinfinite polygon with successive vertices (t^ν). Equivalently, we may define $S_1(x)$ by setting

(2.2) $$S_1(x) = 1 + (t - 1)x \quad \text{if} \quad 0 \leq x \leq 1 ,$$

and extend its definition to all real x by the functional equation

(2.3) $$S_1(x + 1) = tS_1(x) \quad (x \in \mathbb{R}) .$$

Starting from $S_1(x)$, we define the exponential Euler spline

(2.4) $$S_n(x) = S_n(x;t) \quad (x \in \mathbb{R},\ n = 2,3,\ldots)$$

recursively by the averaging operation

(2.5) $$S_n(x) = \int_{x-1/2}^{x+1/2} S_{n-1}(u)du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u)du \quad (n = 2,3,\ldots) .$$

It does seem remarkable that starting from the rough polygon of $S_1(x)$, the repeated smoothing operation

$$\int_{x-1/2}^{x+1/2} (\cdot)du \Big/ \int_{-1/2}^{1/2} (\cdot)du$$

should produce the progressively smoother sequence of interpolants $S_n(x)$. Also that our formula (2.5) does not depend on t explicitly.

In §3 we state the properties of $S_n(x)$, which will also show the validity of their definition (2.5). These properties are established in §§4 and 5.

3. Validity of the definition of the $S_n(x)$ and their properties. Their definition is evidently valid in the case that $t > 0$, for $S_1(x) > 0$ ($x \in R$), and by induction we see that

$$S_n(x) > 0 \quad \text{for all } x \text{ and } n \quad (t > 0) .$$

We loose no generality by assuming in §§ 3, 4, and 5, that in (1.4) we have

$$0 < \alpha < \pi . \tag{3.1}$$

In this case the following propositions will be established.

I. We have

$$I_n(x) := \int_{x-1/2}^{x+1/2} S_n(u)du \neq 0 \quad \text{for} \quad x \in R, \quad n = 1,2,\dots . \tag{3.2}$$

This justifies our definition (2.5) since no denominators can vanish.

II. $S_n(x)$ satisfies the functional equation

$$S_n(x+1) = tS_n(x) \quad \text{for} \quad x \in R, \quad n = 1,2,\dots . \tag{3.3}$$

III. Along the arc

$$A_{n,x} : z = S_n(u) \quad (x - \tfrac{1}{2} \leq u \leq x + \tfrac{1}{2}) \tag{3.4}$$

the argument of $z = S_n(u)$ is steadily increasing by the amount

$$\arg S_n(x + \tfrac{1}{2}) - \arg S_n(x - \tfrac{1}{2}) = \alpha, \qquad \text{(See Figure 1)} . \tag{3.5}$$

The tangent vector $S_n'(u)$ is steadily turning counter-clockwise along $A_{n,x}$ from $S_n'(x - \frac{1}{2})$ to $S_n'(x + \frac{1}{2})$, such that

$$\arg S_n'(x + \tfrac{1}{2}) - \arg S_n'(x - \tfrac{1}{2}) = \alpha . \tag{3.6}$$

Denoting by T the intersection of the lines carrying the vector $S_n'(x \pm \frac{1}{2})$, the four points

$$O,\ S_n(x + \tfrac{1}{2}),\ T,\ S_n(x - \tfrac{1}{2}) \quad \text{are on a circle } \Gamma . \tag{3.7}$$

The arc $A_{n,x}$ is contained in the quadrilateral Q of the four points (3.7).

$$\text{The arc } A_{n,x} \text{ is convex (Figure 1).} \tag{3.8}$$

IV. For all n

$$S_n(x) \in \begin{cases} S_n & \text{if } n \text{ is odd} , \\ S_n^* & \text{if } n \text{ is even} . \end{cases} \tag{3.9}$$

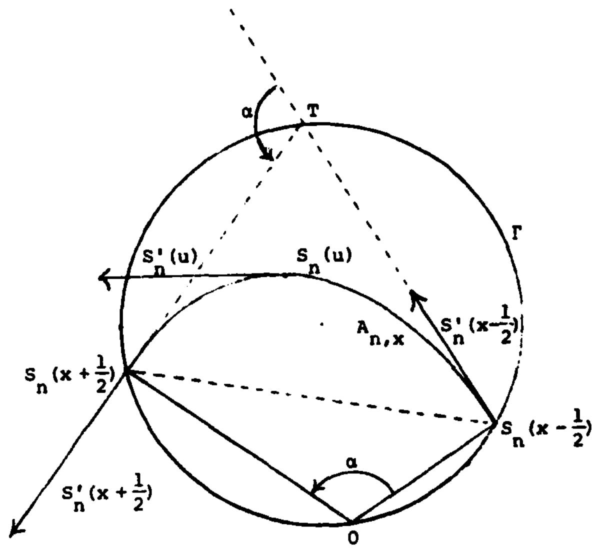

Figure 1

V. The limit relation

$$\lim_{n\to\infty} S_n(x) = t^x = |t|^x e^{i\alpha x} \tag{3.10}$$

holds uniformly in x in every finite interval of $\mathbb{R}$.

VI. For the special case so far excluded when

$$t = -1 \tag{3.11}$$

propositions II and IV remain valid, while I is to be replaced by

$$\int_{-1/2}^{1/2} S_n(u;-1)du > 0 \qquad (n = 1,2,\dots) , \tag{3.12}$$

and V by

$$\lim_{n\to\infty} S_n(x;-1) = \cos \pi x . \tag{3.13}$$

4. Proofs of propositions I, II, III, IV. All these propositions are evident if $n = 1$ in view of (2.3), which implies that

$$\text{triangle } (0, S_1(x - \tfrac{1}{2}), S_1(x + \tfrac{1}{2})) \text{ is similar to triangle } (0,1,t) .$$

Notice also that if $n = 1$ the arc $A_{1,x}$ of (3.4) is identical with the union of the two sides of the quadrilateral Q of Figure 1 that meet in T.

Assume that I, II, III, are already established for all values up to and including $n - 1$ and let us prove them for the value n.

Re II. Using the notation of (3.2), we have

$$S_n(x + 1) = \int_{x+\frac{1}{2}}^{x+\frac{3}{2}} S_{n-1}(u)/I_{n-1}(0) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} S_{n-1}(u + 1)du/I_{n-1}(0) \tag{4.1}$$

$$= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} tS_{n-1}(u)du/I_{n-1}(0) = tS_n(x) .$$

Re III. The relation (3.5) follows immediately from (3.3) by replacing x by $x - \frac{1}{2}$. The remaining statements are obtained as follows: In (2.5), we replace x by u and differentiate with respect to u obtaining

$$S_n'(u) = \{S_{n-1}(u + \tfrac{1}{2}) - S_{n-1}(u - \tfrac{1}{2})\}/I_{n-1}(0)$$
$$= \{tS_{n-1}(u - \tfrac{1}{2}) - S_{n-1}(u - \tfrac{1}{2})\}/I_{n-1}(0)$$

and finally

$$S_n'(u) = c \cdot S_{n-1}(u - \tfrac{1}{2}), \quad \text{where } c = (t - 1)/I_{n-1}(0) \quad (\neq 0) . \tag{4.2}$$

Since $S_{n-1}(u - \frac{1}{2})$ turns counter-clockwise (c.-cl.) by the angle α as u increases from $x - \frac{1}{2}$ to $x + \frac{1}{2}$, we see that (4.2) proves that $S_n'(u)$ turns c.-cl. by the angle α, hence (3.6) holds. This already proves all our statements for n, as summarized in our Figure 1.

Re I. Let x' be such that $x - \frac{1}{2} < x' < x + \frac{1}{2}$ and

$$\arg S(x') = \tfrac{1}{2} \{\arg S_n(x + \tfrac{1}{2}) + \arg S_n(x - \tfrac{1}{2})\} .$$

It follows from (3.5) that

(4.3) $$-\frac{\alpha}{2} \leq \arg S_n(u) - \arg S_n(x') \leq \frac{\alpha}{2} \quad (x - \frac{1}{2} \leq u \leq x + \frac{1}{2}) .$$

Dividing (3.2) by $S_n(x') = |S_n(x')|\exp(i \arg S_n(x'))$ we obtain

$$I_n(x)/S_n(x') = \int_{x-1/2}^{x+1/2} |S_n(u)/S_n(x')| e^{i(\arg S_n(u) - \arg S_n(x'))} du ,$$

and from (4.3) we conclude that

$$\mathrm{Re}\{I_n(x)/S_n(x')\} \geq \int_{x-1/2}^{x+1/2} |S_n(u)/S_n(x')| \cdot \cos \frac{\alpha}{2} du > 0 .$$

This proves (3.2).

Re IV. This is evident by induction from (2.5) because the limits of integration are $x \pm \frac{1}{2}$, in view of the general properties of cardinal splines. It likewise follows by induction that $S_n(x) \in C^{n-1}(\mathbb{R})$ for all n.

5. Proof of proposition V. We write

$$\gamma = \log t = \log|t| + i\alpha, \quad \text{hence} \quad t = e^{\gamma}, \tag{5.1}$$

and define the new function $\omega_n(x)$ by setting

$$S_n(x) = t^x \omega_n(x) \qquad (x \in \mathbb{R}). \tag{5.2}$$

From (3.3) we have that

$$S_n(x+1) = t^{x+1}\omega_n(x+1) = tS_n(x) = tt^x\omega_n(x),$$

hence $\omega_n(x+1) = \omega_n(x)$, showing that $\omega_n(x)$ is periodic of period 1. Using (2.2) let us find the Fourier series of

$$\omega_1(x) = (1 + (t-1)x)e^{-\gamma x} \quad \text{if} \quad 0 \leq x \leq 1. \tag{5.3}$$

Since $\omega_1(x) \in C(\mathbb{R})$ its Fourier series should converge absolutely.

For its Fourier coefficients we find after one integration by parts that

$$a_\nu = \frac{1}{2\pi}\int_0^1 \omega_1(x)e^{-2\pi i\nu x}dx = \frac{(t-1)^2}{2\pi t}\frac{1}{(\gamma + 2\pi i\nu)^2}$$

and so

$$S_1(x) = e^{\gamma x}\omega_1(x) = \frac{(t-1)^2}{2\pi t}\sum_{-\infty}^{\infty}\frac{e^{(\gamma+2\pi i\nu)x}}{(\gamma + 2\pi i\nu)^2}. \tag{5.4}$$

Next we observe that

$$\int_{x-1/2}^{x+1/2} e^{(\gamma+2\pi i\nu)u}du = (-1)^\nu \frac{e^{\gamma/2} - e^{-\gamma/2}}{\gamma + 2\pi i\nu}e^{(\gamma+2\pi i\nu)x}. \tag{5.5}$$

Performing the operation $\int_{x-1/2}^{x+1/2}$ $n-1$ times on (5.4), and dividing the result by its value at $x = 0$ to enforce $S_n(0) = 1$, we obtain the expansion

$$S_n(x) = \sum_{-\infty}^{\infty}\frac{(-1)^{\nu(n-1)}}{(\gamma + 2\pi i\nu)^{n+1}}e^{(\gamma+2\pi i\nu)x} \Big/ \sum_{-\infty}^{\infty}\frac{(-1)^{\nu(n+1)}}{(\gamma + 2\pi i\nu)^{n+1}}. \tag{5.6}$$

Since $t^x = e^{\gamma x}$ we find for $\omega_n(x)$ of (5.2) the expansion

$$(5.7) \qquad \omega_n(x) = \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n-1)}}{(\gamma + 2\pi i\nu)^{n+1}} e^{2\pi i\nu x} / \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n+1)}}{(\gamma + 2\pi i\nu)^{n+1}} .$$

Let us prove that

$$(5.8) \qquad \lim_{n\to\infty} \omega_n(x) = 1 .$$

Proof. We multiply both terms of the fraction (5.7) by γ^{n+1}; except for the two terms for $\nu = 0$, which are $= 1$ in both series, all other terms are in absolute $= |\gamma/(\gamma + 2\pi i\nu)|^{n+1}$ $(\nu \neq 0)$, except for a constant factor. Writing $\rho := \log|t|$ we have $\gamma = \rho + i\alpha$ and so

$$|\gamma/(\gamma + 2\pi i\nu)|^2 = |(\rho + i\alpha)/(\rho + i(\alpha + 2\pi\nu))|^2 = \frac{\rho^2 + \alpha^2}{\rho^2 + (\alpha + 2\pi\nu)^2} .$$

Now our condition $-\pi < \alpha < \pi$ implies that

$$(5.9) \qquad |\alpha + 2\pi\nu| \geq 2\pi - |\alpha| > |\alpha| \quad \text{if} \quad \nu \neq 0 ,$$

and so

$$(5.10) \qquad \min_{\nu} |\gamma/(\gamma + 2\pi i\nu)| = \left(\frac{\rho^2 + \alpha^2}{\rho^2 + (2\pi - |\alpha|)^2}\right)^{1/2} = \delta_t < 1 .$$

Moreover $\delta_t > 0$, since ρ and α can not both vanish because $t \neq 1$. But then it is clear that

$$(5.11) \qquad \omega_n(x) = 1 + O(\delta_t^{n+1}) .$$

From (5.2) and (5.11) we conclude that

$$(5.12) \qquad \frac{S_n(x)}{t^x} - 1 = O(\delta_t^{n+1}), \quad \text{uniformly for } x \in \mathbb{R} ,$$

where δ_t is defined by (5.10).

Notice that δ_t approaches 1 if α is close to $\pm\pi$.

On programming the exponential function on a computer. The construction of $s_n(x)$ as given by our equations (1.2), (1.3), (1.4), (1.5) seems eminently suitable to achieve the title of this subsection. Obtaining the polynomial $p_n(x)$ of (1.3) requires the following algorithm:

Determine the $a_\nu = a_\nu(t)$ recursively from the relations

$$1 + \binom{\nu}{1}a_1 + \binom{\nu}{2}a_2 + \cdots + a_\nu = ta_\nu \quad (\nu = 1,2,\ldots) , \tag{5.13}$$

and then

$$p_n(x) = \{x^n + \binom{n}{1}a_1x^{n-1} + \binom{n}{2}a_2x^{n-2} + \cdots + a_n\}/a_n . \tag{5.14}$$

This is particularly convenient if $t = 2$ when (5.1) reduces to

$$a_\nu = 1 + \binom{\nu}{1}a_1 + \cdots + \binom{\nu}{\nu - 1}a_{\nu-1} \quad (\nu = 1,2,\ldots, a_1 = 1) , \tag{5.15}$$

showing that all coefficients a_ν are integers. Besides, the exponential 2^x is in several ways remarkable, since $\Delta 2^x = 2^x$, and among all entire functions that assume integer values for $x = 0,1,2,\ldots,$ 2^x is the one of least growth (Theorem of Hardy and Polya).

If $t = 2$, then the ρ in (5.10) becomes $\rho = \log 2$; also $\alpha = 0$ and (5.10) shows that

$$\delta_2 = \{1 + \frac{4\pi^2}{(\log 2)^2}\}^{-1/2} = .1096 .$$

This shows that $s_n(x) = s_n(x;2)$ approximates 2^x to nearly n decimal places. In [2, page 403] we found that for $t = 2$

$$p_7(x) = (x^7 + 7x^6 + 63x^5 + 455x^4 + 2625x^3 + 11361x^2 + 32781x + 47293)/47293 .$$

Computations show that

$$|2^x - p_7(x)| < 10^{-7} \quad \text{in} \quad 0 \leq x \leq 1 ,$$

and that

(5.16) the error function $2^x - p_7(x)$ changing sign just once .

An algorithm producing 2^x of course allows us to compute $e^x = 2^{x/\log 2}$.

A better approximation of 2^x is obtained if we replace in $[0,1]$ our polynomial $p_7(x)$ by the polynomial $^*p_7(x)$ of best approximation of 2^x, afterwards obtaining the

approximation $*s_7(x)$ for all real x by $*s_7(x+1) = 2*s_7(x)$. However, the new global approximation $*s_7(x)$ so obtained is evidently discontinuous for all integer values of x, while our approximation $s_7(x)$ is in the class $C^6(-\infty,\infty)$.

Of course, the approximation (5.12) deteriorates as we pass from $t = 2$ to larger values of t: Thus we find that

$$\delta_e = (1 + 4\pi^2)^{-1/2} = .1572, \quad \delta_{10} = \left(1 + \frac{4\pi^2}{(\log 10)^2}\right)^{-1/2} = .3441 .$$

The remark (5.16) is not an isolated result, as we have proposition

VII. 1. If $t > 0$, then the two curves of the (x,y)-plane

(5.17) $$\Gamma_n : y = S_n(x;t), \quad \Gamma : y = t^x \quad (x \in R) ,$$

agree at the points $(x,y) = (\nu,t^\nu)$, the slopes of the tangents to Γ_n being in a fixed non-vanishing ratio to the slopes of the corresponding tangents to Γ. It follows that Γ_n crosses Γ in every interval $\nu < x < \nu + 1$.

2. If in (1.4) we have $0 < \alpha < \pi$, then for the two curves in C

(5.18) $$\tilde{\Gamma}_n : z = S_n(x;t), \quad \tilde{\Gamma} : z = t^x \quad (x \in R) ,$$

we have a similar situation: Either $\tilde{\Gamma}_n$ crosses $\tilde{\Gamma}$ in the same direction at all points t^ν, or else $\tilde{\Gamma}_n$ is tangent to $\tilde{\Gamma}$ at all points t^ν, as in Figure 2 where $|t| = 1$.

Proof. We drop the t writing $S_n(x;t) = S_n(x)$, $t^x = S_\infty(x)$. From the relations $S_n'(x+1) = tS_n'(x)$, $S_\infty'(x+1) = tS_\infty'(x)$, we obtain, for $x = 0$, that

$$\frac{S_n'(\nu)}{S_\infty'(\nu)} = \frac{S_n'(0)}{S_\infty'(0)} \quad \text{for all} \quad \nu \in Z .$$

This implies our conclusions in both cases: For the curves (5.17) we obtain the constancy of the ratio of slopes of their tangents, and for the curves (5.18) we have the constancy of the angle between their tangents.

6. Proof of proposition VI, the case of the Euler splines. Now $t = -1$ and we use for $S_n(x) = S_n(x;-1)$ the term Euler splines, omitting the "exponential". These are well known functions which are the subject of Kolmogorov's famous extremum property of the Euler splines (for references see [4]).

Our construction (2.5) is particularly effective in this case and we will actually prove more than VI and establish the known proposition

VIII. The function $S_n(x) = S_n(x;-1)$ has all the properties of the function $\cos \pi x$ concerning symmetries, zeros, signs, and monotonicities.

Proof. These statements are evident for $n = 1$, when $S_1(x) = E_1(x)$ is the linear Euler spline which the author found so useful in a study of billiard ball motions in a cube [5]. If we assume VIII to be true up to and including $n - 1$, then the validity of all parts of VIII for the function

$$S_n(x) = \int_{x-1/2}^{x+1/2} S_{n-1}(u)du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u)du \tag{6.1}$$

become so obvious, that we may practically omit any further discussion. A few remarks will certainly suffice. The identity $S_n(x + 1) = -S_n(x)$ is proved as in (4.1), and it implies the periodicity $S_n(x + 2) = S_n(x)$. All symmetries and sign properties of $S_{n-1}(x)$ are transmitted to $S_n(x)$ by (6.1), e.g. why is $S_n(x)$ decreasing in $[0,1]$? Answer: Because as x increases from 0 to 1, then the numerator of (6.1) visibly decreases, since it drops positive area while adding negative area (a rough diagram helps).

Let us sketch a proof of (3.13). Since $S_1(x) = |1 - 2x|$ if $-1 \leq x \leq 1$, we obtain the Fourier series

$$S_1(x) = \sum_{-\infty}^{\infty} \frac{1}{(2\nu + 1)^2} e^{(2\nu+1)\pi i x} \Big/ \sum_{-\infty}^{\infty} \frac{1}{(2\nu + 1)^2} . \tag{6.2}$$

Notice also that (5.5) for $\gamma = \pi i$ gives

$$\int_{x-1/2}^{x+1/2} e^{(2\nu+1)\pi i x} dx = \frac{2(-1)^\nu}{(2\nu + 1)\pi} e^{(2\nu+1)\pi i x} .$$

Performing the operation $\int_{x-1/2}^{x+1/2}(\cdot)dx$ on both sides of (7.2) $n-1$ times and dividing by the value at $x = 0$ of the result, we obtain that

$$S_n(x) = \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n+1}} e^{(2\nu+1)\pi i x} \Big/ \sum_{-\infty}^{\infty} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n+1}} .$$

Notice that in both series the coefficients of the terms for $\nu = 0$ and $\nu = -1$ become $= 1$. It follows that we may rewrite the last expansion as

$$S_n(x) = \{e^{\pi i x} + e^{-\pi i x} + \sum_{\nu \neq 0,-1} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n+1}} e^{(2\nu+1)\pi i x}\}/\{1 + 1 + \sum_{\nu \neq 0,-1} \frac{(-1)^{\nu(n-1)}}{(2\nu+1)^{n+1}}\}$$

$$= \{2\cos \pi x + 2 \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^{n+1}} \cos(2\nu+1)\pi x\}/\{2 + 2 \sum_{\nu=1}^{\infty} \frac{1}{(2\nu+1)^{n+1}}\} ,$$

which implies that

$$S_n(x) = \cos \pi x + O(\frac{1}{3^{n+1}}) \quad \text{as} \quad n \to \infty .$$

7. A geometric corollary of (2.5) for bounded exponential Euler splines. The only bounded exponentials t^x have $|t| = 1$, hence $t^x = e^{i\alpha x}$. *However, the corresponding* sequence of splines $S_n(x;t)$ are the most attractive among exponential Euler splines (see [3, dedicatory page and pp. 29-32]). The curve $z = S_n(x;t)$ is bounded iff $|t| = 1$, or

(7.1) $$t = e^{i\alpha}, \text{ with } 0 < \alpha < \pi, \text{ say .}$$

From the unicity property of proposition VII (§6) it easily follows that the curve

(7.2) $$\Gamma_{n,\alpha} : z = S_n(x) = S_n(x;e^{i\alpha}) \qquad (-\infty < x < \infty)$$

has all the rotational and ordinary symmetry properties of the polygon $\Gamma_{1,\alpha} : z = S_1(x;e^{i\alpha})$. Moreover, the curve (7.2) is closed iff $t = e^{i\alpha}$ *is a root of* unity. $\Gamma_{n,\alpha}$ is contained in $|z| < 1$, except at the points $t^\nu = e^{i\nu\alpha}$ which it interpolates. The arcs between two such consecutive points bulge out progressively as n increases, and converge to $|z| = 1$ as $n \to \infty$, by proposition V (§3). The knots of $\Gamma_{n,\alpha}$ are on $|z| = 1$ if n is odd, or in the midpoints of the arcs if n is even.

Figure 2 represents three arcs of each of the consecutive curves

(7.3) $$\Gamma_{n-1,\alpha} \text{ and } \Gamma_{n,\alpha} \qquad (n \geq 2) .$$

On these arcs we have marked in Figure 2 the three points

$$S_n(x),\ S_{n-1}(x + \tfrac{1}{2}),\ S_{n-1}(x - \tfrac{1}{2}) .$$

Differentiating our fundamental relation (2.5) we obtain

(7.4) $$S_n'(x) = c_n(S_{n-1}(x + \tfrac{1}{2}) - S_{n-1}(x - \tfrac{1}{2})), \quad \text{where } c_n = 1/\int_{-1/2}^{1/2} S_{n-1}(u)du .$$

We claim that

(7.5) $$c_n > 0 .$$

Proof. If we let $x = 0$ in (7.4), we obtain

(7.6) $$S_n'(0) = c_n(S_{n-1}(\tfrac{1}{2}) - S_{n-1}(-\tfrac{1}{2}))$$

and Figure 2 shows that both $S_n'(0)$ and $S_{n-1}(\frac{1}{2}) - S_{n-1}(-\frac{1}{2})$ become purely imaginary with positive imaginary parts. The reason: $S_n'(0)$ is obviously vertical and pointing upwards, while $S_{n-1}(\frac{1}{2})$ and $S_{n-1}(-\frac{1}{2})$ are the midpoints of the two symmetric arcs meeting at $z = 1$. Now (7.5) becomes clear from (7.6). We have just proved our last proposition

IX. <u>If we think of</u> x <u>as time, then the velocity vector</u> $S_n'(x)$ <u>is parallel to the vector</u> $S_{n-1}(x + \frac{1}{2}) - S_{n-1}(x - \frac{1}{2})$ <u>and their ratio is a positive constant.</u>

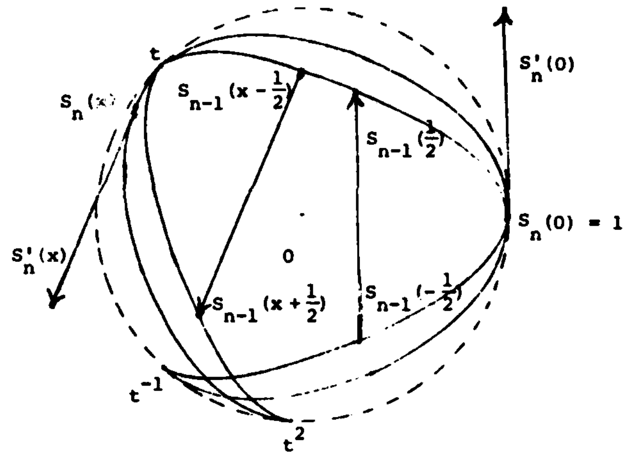

Figure 2

July 1981

Aspen, Colorado

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Starting from the exponential Euler polynomials discussed by Euler in [1], the author introduced in [2] the so-called exponential Euler splines. Here we describe a new approach to these splines. Let t be a constant such that

$$t = |t|e^{i\alpha}, \quad -\pi < \alpha < \pi, \quad t \neq 0, \quad t \neq 1 .$$

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20. ABSTRACT - Cont'd.

Let $S_1(x;t)$ be the cardinal linear spline such that

$$S_1(\nu;t) = t^\nu \quad \text{for all} \quad \nu \in \mathbb{Z} .$$

Starting from $S_1(x;t)$ it is shown that we obtain all higher degree exponential Euler splines recursively by the averaging operation

$$S_n(x;t) = \int_{x-1/2}^{x+1/2} S_{n-1}(u;t)du \Big/ \int_{-1/2}^{1/2} S_{n-1}(u;t)du \qquad (n = 2,3,\ldots) .$$

Here $S_n(x;t)$ is a cardinal spline of degree n if n is odd, while $S_n(x + \frac{1}{2};t)$ is a cardinal spline if n is even. It is shown that they have the properties

$$S_n(\nu;t) = t^\nu \quad \text{for} \quad \nu \in \mathbb{Z} ,$$

$$\lim_{n\to\infty} S_n(x;t) = t^x = |t|^x e^{i\alpha x} .$$

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